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The modular version of Maschke's theorem for normal abelian p-Sylows

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Abstract

If G is a finite group with abelian normal p-Sylow subgroup P, and F is a sufficiently large field of characteristic p, then the group algebra FG can be deformed as an F-algebra to a separable algebra with the same matrix components as in the characteristic zero case.

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Introduction

Maschke's theorem asserts (in modern terminology) that if G is a finite group and F a ring in which the order $|G|$ of G is invertible, then the group algebra FG is separable. That is, the multiplicative map $FG \otimes_F FG \to FG$ splits as a map of FG-bimodules. (Some equivalent conditions are noted later. A finite-dimensional algebra over a field F is separable if and only if, in older terminology, it is absolutely semisimple, i.e., is semisimple and remains so when coefficients are extended to the algebraic closure of F.) If F is an algebraically closed field of characteristic prime to $|G|$, then FG is a direct sum of total matric algebras, the number and the dimensions of which are independent of the characteristic of F. For convenience, we refer to these components as those of the characteristic zero case. By contrast, if F has characteristic p dividing $|G|$, then FG has a non-zero radical.

In this case, however, we have the Donald–Flanigan conjecture: FG can be deformed to a direct sum of total matric algebras which have the same dimensions as in the case

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of characteristic 0. Some special cases have been verified. The case for G abelian is straightforward and was done by Donald and Flanigan themselves [1]. (Here there is, of course, no problem with the dimensions of the matric components.) The problem then seems to have been abandoned until Schaps [7], using Brauer trees, showed that the conjecture holds for G with cyclic p-Sylow subgroups P. When that subgroup is, moreover, normal there is a close connection with the "multicharacteristic" deformation obtained by a finite extension of the integral group ring [8].

In this note we show that if G is a semidirect product $P \rtimes H$ of a normal p-Sylow P and a p'-subgroup H (i.e., one of order prime to p), then (i) a deformation FP can be extended to all of FG iff it is invariant under the actions of H (a virtual tautology), (ii) that when P is abelian there are invariant deformations of FP to separable algebras, and (iii) an extension to FG of such a deformation of G indeed deforms FG to a separable algebra. In particular, the Donald–Flanigan conjecture is therefore verified for the case of an abelian, normal p-Sylow subgroup.

The methods used in this paper, together with a result of Külshammer on the structure of local blocks, have been applied to prove the result for all blocks with abelian normal defect group [5]. In the same paper it is shown that if Broué's conjecture can be proven on the derived equivalences of global and local blocks with abelian defect group, then the Donald–Flanigan conjecture will hold for every block of abelian defect group.

1. Notation

Let G be a finite group, and let p be a prime dividing the order $|G|$ of G. Let F be a field of characteristic p, and denote the group algebra of G over F by FG.

By a formal deformation we mean a deformation of an algebra over the parameter ring $F[[t]]$ of formal power series. By an algebraic deformation we mean a deformation of an algebra over a commutative ring R of finite type over F. Usually, R will be the polynomial ring $F[t]$.

Definition. A *formal (algebraic) p-modular separable deformation* is a formal (algebraic) deformation of FG in which the generic fiber, i.e., the algebra over the quotient field of the parameter ring $F[[t]](R)$ is separable and has a decomposition into matrix blocks identical in degrees to the degrees appearing in the group algebra over an algebraically closed field of non-modular characteristic (i.e., characteristic 0 or prime characteristic not dividing $|G|$).

We are interested in the particular case when G is a semidirect product $P \rtimes H$ of a normal subgroup P (which in applications will be a p-Sylow subgroup of G) and a subgroup H of p' order, i.e., of order not divisible by p. If we denote conjugation hrh^{-1} by r^h, then the multiplication in FG is $(r_1h_1)(r_2h_2) = r_1r_2^{h_1}h_1h_2$. This is the skew group algebra of H with coefficients in the ring FP. We want to extend a p-modular deformation of P to a p-modular deformation of G. The crucial concept in this context is the following.

Definition. Let P be a group, and H a group operating on P. Denote the action of $h \in H$ on $r \in P$ by r^h. We say that a formal deformation of FP is H-invariant if for the deformed multiplication

$$\alpha(r_1, r_2) = \sum f_n(r_1, r_2)t^n, \quad r_1, r_2 \in P,$$

we have

$$\alpha(r_1^h, r_2^h) = \sum f_n^h(r_1, r_2)t^n,$$

where $f_n^h(r_1, r_2) = f_n(r_1, r_2)^h$.

This means that the F-vector space automorphism induced on the deformation by an element of H is in fact an F-algebra automorphism. More generally, we will say that any deformation of FP over a commutative ring R is H-invariant if the action of an element h of H is in fact an automorphism of R algebras.

The significance of this concept can be seen in the following observation. Since H has p' order, the subalgebra FH of FG is separable. Therefore, any deformation of FG is equivalent to a deformation in which the multiplication of FH is unchanged, and the deformation has trivial cohomology as a deformation of FH modules. This is a trivial corollary of the general theory and the fact that the cohomology of a separable algebra in positive dimension vanishes identically. Thus, if $*$ denotes a deformed multiplication in FG, with $r \in P$ and $h \in H$, we will have

$$r * h = rh. \tag{1}$$

Consequently, to use (1) to extend a deformation of FP to all of FG (on FH) we need precisely that the deformation of FP is H-invariant, so that the deformation can be defined as the skew group ring of H with coefficients in the deformation of FP.

1. The extension theorem

Lemma. *Suppose that a finite group H operates as automorphisms of an F-algebra A with multiplication α, and that $\tilde{A}$ with multiplication $\tilde{\alpha}$ is a deformation of A with the property that $\tilde{\alpha}(a^h, b^h) = \tilde{\alpha}^h(a, b)$, i.e., that the deformation is invariant under H. Then $\tilde{\alpha}$ can be extended to a deformation of all the skew group algebra AH by setting*

$$\tilde{\alpha}(a_1 h_1, a_2 h_2) = \tilde{\alpha}(a_1, a_2^{h_1})h_1 h_2,$$

the resulting deformation being just the twisted group algebra $\tilde{A}H$.

Proof. The H invariance implies that H still operates as automorphisms of $\tilde{A}$. The twisted group algebra $\tilde{A}H$ is thus a well-defined associative algebra which reduces to AH in the distinguished fiber.

Theorem. *Let P, H be subgroups of a finite group G such that $G = P \rtimes H$, and suppose that H has p' order, i.e., p does not divide $|H|$. If $A = FP$ has an H-invariant p-modular separable deformation, then $B = FG$ has a p-modular separable deformation.*

Proof. Let $\tilde{A}$ be the H invariant p-modular separable deformation of A. By the previous lemma we know that the twisted group ring $\tilde{A}H$ is a p-modular deformation of $FPH = FG$. It remains only to show that $\tilde{B} = \tilde{A}H$ is generically separable. Let R be the parameter algebra over F with F as residue field at the distinguished prime and K as quotient ring. We know that $\bar{A} = \tilde{A} \otimes_R K$ is separable, and we need to show that

$$\bar{B} = \tilde{B} \otimes_R K = (\tilde{A} \otimes_R K)H$$

is separable over K.

We now consider the generalization of separability given in [6, Section 10.8] for a subalgebra $\bar{A}$ of an algebra $\bar{B}$. We want to show that $\bar{B}$ is separable over $\bar{A}$, for which it suffices to show that there is a separability idempotent $e \in \bar{B} \otimes_{\bar{A}} \bar{B}$ such that $\pi e = e\pi$ for any $\pi \in \bar{B}$ and such that multiplication carries e to 1 in $\bar{B}$. Since H has p' order, we define

$$e = 1/|H| \sum h \otimes_{\bar{A}} h^{-1}.$$

The image of e in $\bar{B}$ is $1/|H| \sum hh^{-1} = (|H|/|H|) \times 1 = 1$. To check the equation $\pi e = e\pi$ it suffices to check it on each element $\pi = rh_1$ of G, since these form a basis of $\bar{B}$ over K:

$$\begin{aligned} rh_1 \times e &= (1/|H|) \sum rh_1(h \otimes_{\bar{A}} h^{-1}) \\ &= (1/|H|) \sum h_1 h r^{(h_1 h)^{-1}} \otimes_{\bar{A}} h^{-1} \\ &= (1/|H|) \sum h_1 h \otimes_{\bar{A}} r^{(h_1 h)^{-1}} h^{-1}) \\ &= (1/|H|) \sum (h_1 h \otimes_{\bar{A}} (h_1 h)^{-1})) r h_1 h h^{-1}) \\ &= e \times rh_1. \end{aligned}$$

Thus, $\bar{B}$ is separable over $\bar{A}$. Since $\bar{A}$ is separable over K, we get $\bar{B}$ separable over K, by transitivity of separability [6, Section 10.8], as required.

2. Normal abelian p-Sylow subgroups

We now consider a special case in which we already know that the p-group has a separable deformation – the case of P abelian. In order to establish the p-modular version of Maschke's theorem for normal abelian p-groups, it now suffices to show that for any automorphism group H of p' order, the deformation of P can be chosen H-invariant.

Theorem. *If P is an abelian group and H is a p'-group acting on P, then P has an H-invariant separable deformation.*

Proof. By the classification theorem for abelian groups, P is a direct product of cyclic groups of prime power order. Collecting together all cyclic groups of the same order, we can write

$$P = P_1 \times \cdots \times P_s,$$

where

$$P_i = C_{p^i} \times \cdots \times C_{p^i}.$$

Given a p'-group of operators H, it is possible to choose the presentation so that each P_i is mapped into itself [4, p.280].

The group algebra of a direct product is the tensor product of the group algebras

$$FP \xrightarrow{\sim} \otimes FP_i.$$

The tensor product of deformations of the FP_i is a deformation of FP. The tensor product of separable algebras is separable [6, Section 10.5].

Thus, we are reduced to proving the theorem where $P = C_q \times \cdots \times C_q$, r times, where $q = p^n$ is a prime power. As above, the group algebra of the tensor product is the r-fold tensor product of the modular group algebra of C_q, which is $F[x]/x^q$. Letting $x_1, \ldots, x_r$ be generators of the various cyclic groups, we therefore have

$$FP = F[x, \ldots, x_r]/(x_1^q, \ldots, x_r^q).$$

We now make one further reduction to the case when F is the finite field $\mathbb{F}_q$. If we can construct an H-invariant p-modular separable deformation of $\mathbb{F}_q P$, then we can make a simple extension of scalars to any field containing $\mathbb{F}_q$. The finite field $\mathbb{F}_{p^n}$ is contained in $\mathbb{F}_{p^m}$ if m is a multiple of n. Thus, in order to find an H-invariant p-modular separable deformation for the original p-group P, which was a product of cyclic subgroups of different orders, we take the least common multiple m of all the exponents in the orders of the cyclic factors, and make an extension of scalars by a field F containing $\mathbb{F}_{p^m}$.

Thus, it will suffice to prove the theorem for $P = C_q \times \cdots \times C_q$ and $F = \mathbb{F}_q$. Let J be the radical of FP. Since H acts as automorphisms of P, FP is a module over FH. Furthermore, J and J^2 must be submodules, since any automorphisms preserves the radical. FH is separable and J^2 is a submodule of J, so J must contain a complementary H-submodule N isomorphic to J/J^2, and thus of dimension r. Let $y_1, \ldots, y_r$ be a basis for N; the mapping $\phi : FP \to FP$ given by $\theta(z) = z^q$ is an F-algebra homomorphism because $c^q = c$ for all $c \in F = \mathbb{F}_q$ and $(z + w)^q = z^q + w^q$. Since $(x_1, \ldots, x_r)$

and $(y_1,\ldots,y_r)$ generate the same ideal J of FP, we conclude that $(x_1^q,\ldots,x_r^q)$ and $(y_q^q,\ldots,y_r^q)$ also generate the same ideal, and thus $FP \xrightarrow{\sim} F[y_1,\ldots,y_r]/(y_1^q,\ldots,y_r^q)$.

We now let t be an indeterminate, and construct a deformation of FP:

$$\tilde{A} = F[t][y_1,\ldots,y_r]/(y_1^q - t^{q-1}y_1,\ldots,y_r^q - t^{q-1}y_r).$$

We claim that this deformation is H-invariant and separable. We begin with H-invariance. The deformation is flat, being the tensor product of flat deformations

$$F[t][y_i]/(y_i^q - t^{q-1}y_i).$$

There is a basis consisting of all monomials with maximum degree $q-1$ in each variable. Since the F-vector space W generated by $y_1,\ldots,y_n$ is an FH-module, for each $h \in H$ we have

$$y_i^h = a_{i1}y_1 + \cdots + a_{ir}y_r, \quad i = 1,\ldots,r$$

and the matrix $[a_{ij}]$ is nonsingular. Thus, H induces a degree preserving FH algebra automorphism of $F[t][y_1,\ldots,y_r]$ into itself. This automorphism carries the ideal generated by the $y_i^q - t^{q-1}y_i$ into itself, since, using the fact that $a_{ij}^q = a_{ij}$, we have

$$\begin{aligned}(y_i^q - t^{q-1}y_i)^h &= (y_i^h)^q - t^{q-1}y_i^h \\ &= (a_{i1}y_1 + \cdots + a_{ir}y_r)^q - t^{q-1}(a_{ij}y_i + \cdots + a_{ir}y_r) \\ &= (a_{i1}^q y_1^q + \cdots + a_{ir}^q y_r^q) - t^{q-1}(a_{ij}y_1 + \cdots + a_{ir}y_r) \\ &= a_{i1}(y_1^q - t^{q-1}y_1) + \cdots + a_{ir}(y_r^q - t^{q-1}y_r)\end{aligned}$$

and the matrix (a_{ij}) is nonsingular. Thus, H actually induces automorphisms of the quotient algebra $\tilde{A}$, as required

As for separability, we first note that for $t \neq 0$, the polynomial $y_i^q - t^{q-1}y_i$, splits completely into q distinct linear factors $(y_i - tc)$, as c runs over all q elements of the finite field $\mathbb{F}_q$. Thus, each $F[t][y_i]/(y_i^q - t^{q-1}y_i)$ is generically separable. Since, as mentioned above, the tensor product of separable algebras is separable, we conclude that $\tilde{A}$ is generically separable, and gives the required H-invariant p-modular separable deformation.

Example. The Klein 4-group $C_2 \times C_2$ has automorphism group isomorphic to S_3. Here $p = 2$, and the only p'-subgroup of S_3 is C_3, the corresponding automorphism being given by cyclic permutation of the nonidentity elements of K_4.

We set $F = \mathbb{F}_2$. Our problem is to find a C_3-invariant p-modular deformation of FK_4 to F^4. Let a and b be generators, and $c = ab$, so that the action of C_3 is $a \to b \to c \to a$. As described in the proof of the theorem, we must find an H-submodule complementary to $J^2 = (e+a+b+c)$. The module generated by $x = a+b$ and $y = b+c$ will be satisfactory, since the action of H is given by

$$a+b \to b+c \to c+a \to a+b,$$

$$x \to y \to x+y \to x.$$

The group algebra FP in this basis is still given by

$$F[x,y]/(x^2,y^2).$$

The deformation given by the deformed relations is then

$$F[t][x,y](x^2 - tx, y^2 - ty).$$

If we substitute a nonzero element s of F for t, this ring is isomorphic to F^4. The four points correspond to the values 0 and t for x and y.

We note that this deformation is not the only separable deformation, or even the only "natural" deformation.

To see other possibilities, suppose F is enlarged to include a cubed root ω of 1. Since C_3 is an abelian group, the irreducible representations on FK_4 are all one dimensional, with eigenvalues which are cubed roots of one $1,\omega,\omega^2$. The eigenspaces are:

(a) $\langle e, a+b+c\rangle$ for $\lambda = 1$,

(b) $\langle z_1 = a+\omega b+\omega^2 c\rangle$ for $\lambda = \omega$,

(c) $\langle z_2 = a+\omega^2 b+ac\rangle$ for $\lambda = \omega^2$.

Let J be the radical and let $z_3 = e+a+b+c = (e+a)(e+b)$. Since $J^2 = \langle z_3\rangle = \langle e+a+b+c\rangle$, the two eigenvectors z_1 and z_2 are linearly independent generators of the radical J, and thus

$$FK_4 \xrightarrow{\sim} F[z_1,z_2]/(z_1^2,z_2^2).$$

Let θ be the generator of C_3, acting on all of $F[z_1,z_2]$ via $\theta(z_1) = \omega z_1$, and $\theta(z_2) = \omega^2 z_2$. Since $\theta(z_1^2) = \omega^2 z_1^2$ and $\theta(z_2^2) = \omega^2 z_2^2$, we get a C_3 invariant deformation by taking the ideal $(z_1^2 - tz_2, z_2^2 - tz_1)$, because z_1^2 and z_2 have the same eigenvalue ω^2, and z_2^2, z_1 have the same eigenvalue ω. This corresponds to a multiplication

$$\alpha(z_1,z_1) = tz_2, \quad a(z_i,z_3) = t^2 z_i, \quad i = 1,2,$$

$$\alpha(z_2,z_2) = tz_1, \quad a(z_3,z_i) = t^2 z_i, \quad i = 1,2,$$

$$\alpha(z_1,z_2) = z_3, \quad a(z_3,z_3) = t^2 z_3,$$

$$\alpha(z_2,z_1) = z_3.$$

This can be rewritten as $F[z_1]/z_1^4 - t^3 z_1$, which factors completely into a direct sum of $F[z_1]/z_1 - a_i t, \quad a_i = 0,1,\omega,\omega^2$.

Additional note: Although it is not relevant to the extension theorem, it is actually possible to find a first order 2-modular semisimple deformation of FK_4 which is invariant under the entire automorphism group $\mathrm{Aut}(K_4) \xrightarrow{\sim} S_3$. Since the desired deformation is commutative, it is necessary to find a first-order deformation over $F[\varepsilon]$, $\varepsilon^2 = 0$,

$$F[\varepsilon][x, y]/(x^2 - \varepsilon(a_1x + b_1y + c_1xy), y^2 - \varepsilon(a_2x + b_2y + c_2xy)).$$

Subjecting this deformation to all possible automorphisms produces only one possibility, up to multiplication of ε by a constant:

$$F[\varepsilon][x, y]/(x^2 - \varepsilon(x + xy), y^2 - \varepsilon(y + xy)).$$

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